Mathematics: analysis and approaches

Higher level

Paper 3

Name

Date: ____________________

1 hour

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

exam: 3 pages

Answer all questions on separate answer paper / booklet. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 29]

The probability density function f is defined as follows where $a \in \mathbb{R}$.

$$f(x) = \begin{cases} 0, & x < 0 \\ \dfrac{\sin x}{4}, & 0 \le x \le \pi \\ a(x-\pi), & \pi < x \le 2\pi \\ 0, & x > 2\pi \end{cases}$$

(a) Sketch the graph of $y = f(x)$. [2]

(b) Find $\mathrm{P}(X \le \pi)$. [2]

(c) Show that $a = \dfrac{1}{\pi^2}$. [3]

(d) Write down the median of X. [1]

(e) Show that the mean of X is $\dfrac{13\pi}{12}$. [6]

(f) Calculate the variance of X. [3]

(g) Find $\mathrm{P}\left(\dfrac{\pi}{2} \le X \le \dfrac{3\pi}{2}\right)$. [2]

(h) Given that $\dfrac{\pi}{2} \le X \le \dfrac{3\pi}{2}$, find the probability that $\pi \le X \le 2\pi$. [4]

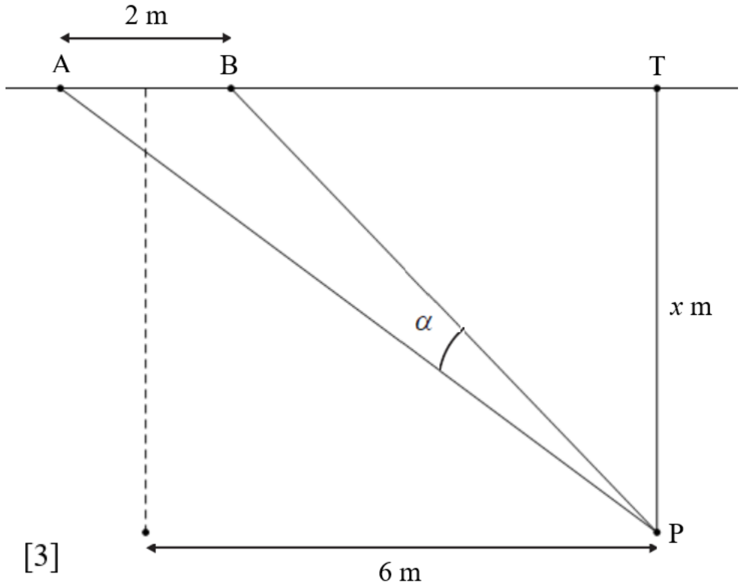
(i) Consider the function g defined as follows where $c \in \mathbb{R}$.

$$g(x) = \begin{cases} 0, & x < 0 \\ \dfrac{\sin x}{c}, & 0 \le x \le \pi \\ c(x-\pi), & \pi < x \le 2\pi \\ 0, & x > 2\pi \end{cases}$$

Show that there is no value of c such that g is a probability density function. [6]

2. [Maximum mark: 26]

Points A, B and T lie on a line on an indoor soccer field. The goal, $[AB]$, is 2 metres wide. A player situated at point P kicks a ball at the goal. $[PT]$ is perpendicular to (AB) and is 6 metres from a parallel line through the centre of $[AB]$. Let PT be x metres and let $\alpha = \text{APB}$ measured in degrees. Assume that the ball travels along the floor.



(a) Find the value of α when $x = 10$. [3]

(b) Applying an appropriate trigonometric identity, show that $\tan\alpha = \dfrac{2x}{x^2+35}$. [3]

The maximum value for α occurs when $\tan\alpha$ is a maximum.

(c) (i) Find $\dfrac{\text{d}}{\text{d}x}(\tan\alpha)$.

(ii) Hence, find the value of α such that $\dfrac{\text{d}}{\text{d}x}(\tan\alpha) = 0$.

(iii) Find $\dfrac{\text{d}^2}{\text{d}x^2}(\tan\alpha)$ and hence show that the value of α never exceeds $10°$. [10]

(d) Find the set of values for x for which $\alpha \geq 7°$. [3]

Consider that, instead of being 6 metres, the distance from $[PT]$ to the parallel line through the centre of $[AB]$ is y metres.

(e) (i) Show that $\tan\alpha = \dfrac{2x}{x^2+y^2-1}$. [3]

(ii) Show that $x = \sqrt{y^2-1}$ when α is a maximum. [4]